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ROCKET BOOSTER CONTROL

SECTION 13

AN APPLICATION OF THE QUADRATIC PENALTY FUNCTION CRITERION TO THE DETERMINATION OF A LINEAR CONTROL FOR A FLEXIBLE VEHICLE

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1	Summary
1541-TR 2	Control of Plants Whose Representation Contains Derivatives of the Control Variable
1541-TR 3	Modes of Finite Response Time Control
1541-TR 4	A Sufficient Condition in Optimal Control
1541-TR 5	Time Optimal Control of Linear Recurrence Systems
1541-TR 6	Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
1541-TR 7	Penalty Functions and Bounded Phase Coordinate Control
1541-TR 8	Linear Programming and Bounded Phase Coordinate Control
1541-TR 9	Time Optimal Control with Amplitude and Rate Limited Controls
1541-TR 10	A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
1541-TR 11	A Note on System Truncation
1541-TR 12	State Determination for a Flexible Vehicle Without a Mode Shape Requirement
1541-TR 13	An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
1541-TR 14	Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
1541-TR 15	An Alternate Derivation and Interpretation of the Drift-Minimum Principle
1541-TR 16	A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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AN APPLICATION OF THE QUADRATIC PENALTY FUNCTION CRITERION TO THE DETERMINATION OF A LINEAR CONTROL FOR A FLEXIBLE VEHICLE*

By E. E. Fisher‡

16754

ABSTRACT

A

A linear controller is designed for a typical five-engine flexible rocket booster by use of the quadratic penalty function criterion. Consideration is given to the sensor requirement for controller implementation. It is shown that the optimal controller can be approximated using less sensors than the plant order.

Author

INTRODUCTION

A difficult problem in the application of state vector control theory to the flexible vehicle is that of determining a method of control feedback. Control laws are determined in terms of the components of the state vector. Sensors, however, measure a blend of state vector components. Superficial considerations indicate that a large number of sensors is required in order that all of the "dominant" state vector components can be determined. From these components the feedback is then usually established. This paper will investigate the possibility of implementing a quadratic criterion controller using some minimal number of sensors, that is, using less than the above mentioned large number of sensors.

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* Prepared under contract NASw-563 for the NASA.

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Specifically, it is assumed a finite flexure mode model of the flexible vehicle is given. Such a model is to include all flexure modes which will be of importance for the type of controllers and wind disturbances under consideration. Let n be the order of this model. A linear control law will be determined by use of the quadratic criterion. Methods will be investigated by which the derived controller can be implemented using less than n sensors.

The dynamics of the assumed vehicle are presented first.

VEHICLE DYNAMICS

The dynamical equations of a typical five-engine flexible rocket booster are presented. Consideration is given to rigid body and three flexure degrees of freedom. Fuel sloshing and engine dynamics are not considered. Linear equations with constant coefficients are assumed. The motion considered is in the pitch plane. With recourse to the definitions in Table 1 the equations of motion are equations 1 through 4.

$$\ddot{z} = \frac{F-X}{m}\varphi + \frac{N'}{m}\alpha + \frac{R'}{m}\beta + \sum_{j=1}^{3} d_j\eta_j \qquad \text{Lateral Path Motion} \tag{1}$$

$$\ddot{\varphi} + c_1\alpha + c_2\beta + \sum_{j=1}^{3} e_j\eta_j = 0 \qquad \text{Angular Motion} \tag{2}$$

$$\alpha = \varphi - \frac{\dot{z}}{V_o} \qquad \text{Angular Relationship} \tag{3}$$

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = f_i\beta \qquad i = 1,2,3 \qquad \text{Flexure Motion} \tag{4}$$

Here · represents differentation with respect to time. New variables are defined so that the equations can be rewritten in the standard state vector form. The change of variables is defined by equation 5.

$$\begin{array}{lllll} x_1 = \varphi & x_2 = \dot{\varphi} & x_3 = \alpha & x_4 = \eta_1 & x_5 = \dot{\eta}_1 \\ x_6 = \eta_2 & x_7 = \dot{\eta}_2 & x_8 = \eta_3 & x_9 = \dot{\eta}_3 & \beta = u \end{array} \tag{5}$$

On this transformation the equations of motion become those given by equation 6.

$$\dot{x} = Ax + bu \tag{6}$$

Here x is a column vector with components $x_1, x_2, \ldots x_9$ and A and b are matrices as listed below with a_{ij} and b_i appropriate constants

$$A = \begin{vmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} & 0 & a_{22} & 0 & a_{28} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 & a_{36} & 0 & a_{38} & 0 \\ 0 & 0 & 0 & 0 & a_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{67} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{76} & a_{77} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{98} & a_{99} \end{vmatrix} \qquad b = \begin{vmatrix} 0 \\ b_2 \\ b_3 \\ 0 \\ b_5 \\ 0 \\ b_7 \\ 0 \\ b_9 \end{vmatrix} \tag{7}$$

Of prime concern is the information available from sensors for feedback purposes. If $s_a(y)$ is the output of an accelerometer located at a station y meters from the rocket's booster's tail, it can be shown that $s_a(y)$ will be given by equation 8. (Again the definitions in Table 1 are used).

$$s_a(y) = \frac{R'}{m}\left\{\beta + \sum_{i=1}^{3} Y_i(y_\beta)\eta_i\right\} + \frac{N'}{m}\alpha + (y_{cg} - y)\ddot{\varphi}$$
$$+ \sum_{i=1}^{3} Y_i(y_\beta)\ddot{\eta}_i - \sum_{i=1}^{3}\left[\frac{F}{m} Y_i(y_\beta) - \frac{(F-X)}{m} Y_i(y)\right]\eta_i \qquad (8)$$

If $\ddot{\varphi}$ and $\ddot{\eta}_i$ are replaced in equation (8) by their values as given by equations 2 and 4 there results

$$s_a(y) = \sum_{j=3}^{9} z_{aj}(y)x_j + z_{ao}u \qquad (9)$$

Here the coefficients $z_{aj}(y)$ are functions which depend on y either linearly or through the mode slope functions $Y_i(y)$. It is noticed that for a given fixed sensor location y the output of an accelerometer is a linear non-homogeneous function of the state variables $x_1, x_2, \ldots x_9$. Since the non-homogeneous term $z_{a0}u$ can in principle be subtracted from the sensor output, $s_a(y)$ will henceforth be considered given by equation (9) with z_{a0} set equal to zero.

If $s_r(y)$ is the output of a rate gyro located y meters from the rocket booster's tail, it can be shown that $s_r(y)$ is as given by equation (10).

$$s_r(y) = \dot{\varphi} + \sum_{i=1}^{3} Y_i(y)\dot{\eta}_i \qquad (10)$$

Finally if $s_p(y)$ is the output of a position gyro (angular position) located y meters from the rocket booster's tail, it can be shown that $s_p(y)$ is as given by equation (11).

$$s_p(y) = \varphi + \sum_{i=1}^{3} Y_i(y)\eta_i \qquad (11)$$

The important thing to be noticed is that both rate and position gyros measure linear combinations of the state variables $x_1, x_2, \ldots x_9$. To form a correspondence between the outputs $s_r(y)$, $s_p(y)$, and $s_a(y)$ equation (10) and (11) will be rewritten as given by equation (12).

$$
\begin{aligned}
s_r(y) &= \sum_{j=2,5,7,9} z_{rj}(y) x_j \\
s_p(y) &= \sum_{j=1,4,6,8} z_{pj}(y) x_j
\end{aligned}
\tag{12}
$$

Here the coefficients $z_{rj}(y)$ and $z_{pj}(y)$ are the appropriate functions of y. In what follows any sensor whose output is a linear homogeneous function of the state variables (as the above are) will be called a linear sensor.

If the output of nine linear sensors is given, and if the associated matrix (which depends on the locations of the nine sensors, as well as their type) is non-singular, then the nine time functions $x_1, x_2, \ldots x_9$ can be expressed as a linear combination of the nine sensor outputs. Thus any control law (a rule which determines u as a function of $x_1, \ldots x_9$ is called a control law) can, in this case, in principle, be constructed. Later, consideration is given to the problem of implementing a given linear control law using less than nine sensors.

This suffices to introduce plant and sensor dynamics; thus attention is turned to controller design.

DESIGN OF VEHICLE CONTROLLER BY THE QUADRATIC CRITERION

The quadratic criterion is discussed with comments about its applicability to the flexible vehicle problem. The design of a controller for the given vehicle is then presented.

THE QUADRATIC PENALTY FUNCTION CRITERION

Let a system be governed by the state equation

$$\dot{x} = Ax + bu \tag{13}$$

where x is an n-vector which represents the state of the system, A is a constant n by n matrix, b is a constant n vector, and u is a scalar. Consider the problem of determining a control law

$$u = k'x \tag{14}$$

Where k is an n dimensional constant vector and $'$ indicates transpose. For a given positive definite symmetric matrix Q a control law $u = k'x$ is desired which minimizes V(t) given by equation (15).

$$2V(t) = \lim_{T \to \infty} \int_t^T (x'Qx + u^2)d\tau \tag{15}$$

Here t is the present time. Kalman (reference 3) has shown that under appropriate conditions of controllability there exists a unique positive definite matrix P which satisfies the matrix equation

$$A'P + PA - Pbb'P + Q = 0 \tag{16}$$

such that the control $u = k'x$ with k given by

$$k = - Pb \tag{17}$$

is the unique optimum control.

Designing via the quadratic penalty criterion is an iterative process which starts with estimating the weighting matrix Q on the basis of physical considerations. Equation (16) is then solved for the associated P matrix. The gain vector k is established by equation (17) and the closed loop system is examined for physical characteristics. This leads to the need for a re-adjustment of the weighting matrix Q and the process is thus iterated.

The exact physical characteristics required are stabilization of the unstable air-frame, small deviation from the desired attitude, small angle of attack to reduce bending loads, bounded engine deflection and rates, etc. With the exception of stability, none of these things result automatically from the quadratic criterion. The quadratic criterion does, however, offer an orderly iterative process by which an acceptable controller can be constructed. This is an advantage over some conventional methods.

DESIGN OF A CONTROLLER FOR THE GIVEN VEHICLE

Numerical values for the matrices A and b are given in Table 2. The units used are meters, radians, and seconds. For purposes of comparison the characteristic roots $\lambda_1, \lambda_2, \ldots \lambda_9$ of the matrix A are presented in Table 3.

High order in the differential equations of the plant as well as a natural interest in rigid body motion motivated a study of a fictitious rigid body governed by equations (18). A controller was designed for this rigid body by a series of adjustments, and re-adjustment of the matrix $Q^{(1)}$.

$$\dot{x}^{(1)} = A^{(1)}x^{(1)} + b^{(1)}u^{(1)}$$

$$x^{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 0 \\ b_2 \\ b_3 \end{pmatrix}, \quad K^{(1)} = \begin{pmatrix} k_1^{(1)} \\ k_2^{(1)} \\ k_3^{(1)} \end{pmatrix} \qquad (18)$$

and $u^{(1)} = k^{(1)}x^{(1)}$

After several iterations a suitable weighting matrix $Q^{(1)}$ with corresponding matrix $P^{(1)}$ were arrived at. The resulting closed loop characteristic roots were used as a criterion for the final selection of $Q^{(1)}$. That is, the desired weighting matrix $Q^{(1)}$ was chosen on the basis of the characteristic roots of the matrix $A^{(1)} + b^{(1)}k^{(1)}$ which resulted from $Q^{(1)}$. Characteristic roots and associated gains are presented in Table 4. Weighting matrices used were diagonal and of the form $Q^{(1)} = [\delta_{ij}\ q_i^{(1)}] = [q_{ij}^{(1)}]$, $q_i^{(1)} > 0$. The controller listed as case 4 was considered the desired rigid body controller.

A fifth order system consisting of a fictitious rigid body with one flexure mode was considered next. Its dynamics is governed by equations (19).

$$\dot{x}^{(2)} = A^{(2)}x^{(2)} + b^{(2)}u^{(2)}$$

$$x^{(2)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \quad b^{(2)} = \begin{bmatrix} 0 \\ b_2 \\ b_3 \\ 0 \\ b_5 \end{bmatrix}, \quad k^{(2)} = \begin{bmatrix} k_1^{(2)} \\ k_2^{(2)} \\ k_3^{(2)} \\ k_4^{(2)} \\ k_5^{(2)} \end{bmatrix} \qquad (19)$$

$$\text{and } u^{(2)} = k^{(2)}\ x^{(2)}$$

Here the weighting matrix $Q^{(2)} = [\delta_{ij}q_i^{(2)}]$ was used, where the elements $q_1^{(2)} q_2^{(2)}$ and $q_3^{(2)}$ where equal respectively to $q_1^{(1)}, q_2^{(1)}$, and $q_3^{(1)}$; and $q_4^{(2)}$ and $q_5^{(2)}$ were chosen quite small. This building process was motivated by difficulties in solving equation (16) for high order systems. Although various procedures were tried in its solution, integration of the Riccati matrix differential equation (given by equation (20)) to a steady state was finally adopted as a method of solution*

$$\frac{-dP}{dt} = A\ P + AP - Pbb\ P + Q \tag{20}$$

An initial estimate of $P^{(2)}$ (the solution of equation (16) for $A^{(2)}$,$B^{(2)}$ and $Q^{(2)}$) was a matrix with its fourth and fifth rows and columns set equal to zero and its remaining nine elements set equal to the corresponding ones in $P^{(1)}$(the solution of equation (16) for $A^{(1)}$,$B^{(1)}$, and $Q^{(1)}$). This was used as an initial condition for the Riccati equation which converged from there quite rapidly. The process was extended to seventh and finally ninth order systems for a variety of cases. Some of the results for ninth order systems are presented in Table 5. In each case it was found that the characteristic roots and associated gains changed very little as the system was built up from the rigid body problem. Table 6 indicates the progress of such a process for the characteristic roots and associated gains. Table 7 indicates the progress for the P matrices. Case 1 corresponds to a rigid body problem with weighting q_1 = .1, q_2 = .05, and q_3 = .5. Each sucessive case had two more variables added (i.e., one more flexure mode) with additional weighting equal to .0001.

It is noted that the weighting factor w defined in Table 5

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* It is easily seen that if P(t) is a matrix which satisfies equation (20) and if P(t) has reached a steady state equal to P, a constant, then P satisfies equation (16). Kalman (reference 3) shows that such a steady state is indeed a unique positive definite solution of equation (16).

has little effect on the closed loop flexure frequencies and that its effect on closed loop damping goes as the square root of w. Another trend as pointed out by Reynolds and Rynaski (reference 1) is that the coefficients of the characteristic equation increase in magnitude with an increase in the norm of the matrix Q.

Previous experience with the flexible vehicle problem indicates the controller listed as case 4 in Tables 6 and 7 as a reasonable controller. Consideration will be given next to its implementation.

CONSTRUCTION OF THE LINEAR CONTROL LAW GIVEN BY THE QUADRATIC CRITERION

A general linear sensor with location y_i is defined to such that its output $s_i(y_i)$ is given by equation (21).

$$s_i(y_i) = \sum_{j=1}^{9} z_{ij}(y_i)x_j \tag{21}$$

Here the coefficients z_{ij} depend in some fashion on y_i as indicated. It is noted that s_i and x_i are time functions although this dependence is not indicated. Let m sensors of the above nature be stationed at locations $y_1, y_2, \ldots, y_m$ where $m \leq 9$. Let the output of each sensor be multiplied by an as yet undetermined sensor gain γ_i and summed to form a feedback quantity $\tilde{u}$ given by equation (22)

$$\tilde{u} = \sum_{i=1}^{m} \gamma_i s_i = \sum_{i=1}^{m} \gamma_i \left(\sum_{j=1}^{9} z_{ij} x_j \right) \tag{22}$$

If $k_1, k_2, \ldots, k_9$ are the feedback gains given by the quadratic criterion ($k_1, k_2, \ldots, k_9$ are the components of the vector k as previously defined) the desired feedback u is given by equation (23).

$$u = \sum_{j=1}^{9} k_j x_j \tag{23}$$

The quantity $\tilde{u}$ will be equal to u if equation (24) is satisfied.

$$\sum_{i=1}^{m} \gamma_i z_{ij} = k_j \qquad j = 1,2,\ldots,9 \tag{24}$$

This equation can be interpreted as a vector equation in the vectors z_i and k given by equation (25).

$$\begin{aligned} z_i &= (z_{i1}, z_{i2}, \ldots, z_{i9})' \\ k &= (k_1, k_2, \ldots, k_9)' . \end{aligned} \tag{25}$$

The vector equation is then that given by equation (26).

$$\sum_{i=1}^{m} \gamma_i z_i = k. \tag{26}$$

In the event that m is 9 and the vectors z_i are linearly independent (i.e., they span the nine dimensional space) there exist constants γ_i such that equation (26) is true for any given vector k. This is a statement of the fact that nine independent* sensors serve to determine the nine components of the state vector. (This result was shown by Harvey in reference 2). If m is less than nine equation (26) will not be satisfied for arbitrary k, but only those k which lie in the space spanned by

---

*For linear sensors a set of sensors will be said to be independent if the associated vectors z_i are linearly independent. It is tacitly assumed that this can be achieved by the correct variety of sensors at the correct sensor locations.

$z_1, z_2, \ldots, z_m$. Since k is given by the quadratic criterion and thus fixed, solution of equation (26) with m less than nine involves selecting the z_i in such a way that they span a space which includes the vector k. Because the vectors z_i each depend on the position of the i^{th} sensor y_i there is some hope that, with proper sensor positioning, equation (26) can be solved for m less than nine. Because of the physical nature of the problem, exact solutions are not necessary. For instance, the vector $\sum_{i=1}^{m} \gamma_i z_i$ can be considered close to k if the magnitude of their difference vector or its square is small. Its square is therefore introduced as an error quantity E given by equation (27).

$$E = \left(\sum_{i=1}^{m} \gamma_i z_i - k \right)^2 \tag{27}$$

Consideration of the actual problem at hand dictates how the error function E is to be used. It is noted that the vectors $z_i(y_i)$ originate from accelerometers, rate gyros, or position gyros. That is, the m sensors indicated in equation (27) are as yet not determined combination of accelerometers, rate gyros, and position gyros. There are located at certain specific, though as yet undetermined, locations $y_1, y_2, \ldots, y_m$. As is usually the case, the error function E should be minimized with respect to its parameters to yield the best set of parameters. These parameters include the m sensor gains γ_i and the m sensor positions y_i. If E is minimized with respect to the sensor positions y_i, the mathematical minimum may result in certain of the y_i's being at the ends of the booster. This could happen because the variables y_i are defined only for $0 < y_i < \ell$ where

ℓ is the booster length. Also minimization with respect to the y_i's is not a linear problem since the y_i's enter the problem non-linearly through the mode slope functions. For these reasons the error function E will not be minimized with respect to the y_i's. Instead the positions y_i will be assumed given and the error function E will be minimized (for the given y_i s) in terms of the γ_i's. The γ_i's which minimize E for given fixed y_i's are given by the familiar linear least squares normal equations, equations (28).

$$\sum_{j=1}^{m} \gamma_j (z_i \cdot z_j) = z_i \cdot k \qquad i = 1,2,\ldots,m \tag{28}$$

(Here $\cdot$ indicates the vector dot product)
It is noted that for many rocket booster problems it may be advantageous to minimize E with respect to both the y_i's and γ_i's. This is not done here.

Attention is now turned to the calculation at hand, the approximation of the feedback derived. (This is displayed in Tables 6 and 7 as case 4.)

Three basic types of sensors (accelerometers, rate gyros, and position gyros) are assumed. These will have sensor gain vectors z_a, z_r, z_p given by equation (29). The mode shape functions $Y_1(y)$, $Y_2(y)$, and $Y_3(y)$ whose derivatives appear in equation (29) are given by the polynomials in equation (30). The location variable y is assumed to be in the interval (0,100).

$$
\begin{array}{lll}
z_{a1}(y) = 0 & z_{r1}(y) = 0 & z_{p1}(y) = 1 \\
z_{a2}(y) = 0 & z_{r2}(y) = 1 & z_{p2}(y) = 0 \\
z_{a3}(y) = \ 39.28 - 1.138y & z_{r3}(y) = 0 & z_{p3}(y) = 0 \\
z_{a4}(y) = -20.89\ Y_1(y) & z_{r4}(y) = 0 & z_{p4}(y) = Y_1(y) \\
\qquad -.036y - .29.02 & & \\
z_{a5}(y) = -.055 & z_{r5}(y) = Y_1(y) & z_{p5}(y) = 0 \\
z_{a6}(y) = -20.89\ Y_2(y) & z_{r6}(y) = 0 & z_{p6}(y) = Y_2(y) \\
\qquad -.030y - 1685 & & \\
z_{a7}(y) = -.130 & z_{r7}(y) = Y_2(y) & z_{p7}(y) = 0 \\
z_{a8} \quad = -20.89\ Y_3(y) & z_{r8}(y) = 0 & z_{p8}(y) = Y_3(y) \\
\qquad -.027y - 333.9 & & \\
z_{a9} \quad = - .183 & z_{r9}(y) = Y_3(y) & z_{p9}(y) = 0
\end{array}
\tag{29}
$$

$$
\begin{aligned}
Y_1(y) = {} & .1045 \times 10 - .6319 \times 10^{-1}y + .5917 \times 10^{-3}y^2 + .3273 \times 10^{-5}y^3 \\
& -.2429 \times 10^{-6}y^4 + .4599 \times 10^{-8}y^5 - .2156 \times 10^{-10}y^6 \\
y_2(y) = {} & .1349 \times 10 - .7790 \times 10^{-1} - .5233 \times 10^{-2}y^2 \\
& + .4182 \times 10^{-3} - .1115 \times 10^{-4}y^4 + .1575 \times 10^{-6}y^5 \\
& - .1167 \times 10^{-8}y^6 + .3451 \times 10^{-4}y^7 \\
y_3(y) = {} & .1079 \times 10 - .5980 \times 10^{-1} - .1566 \times 10^{-2}y^2 \\
& -.1956 \times 10^{-3}y^3 + .1931 \times 10^{-4}y^4 - .4760 \times 10^{-6}y^5 \\
& + .4577 \times 10^{-8}y^6 - .1527 \times 10^{-10}y^7
\end{aligned}
\tag{30}
$$

Several sensors must be positioned in such a way that the gain vector k can be approximated by a linear combination of sensor outputs. An examination of the sensor gains (equation (29)) indicates at least one accelerometer must be used in order to pick up x_3, the angle of attack; at least one rate gyro must be used to determine pitch rate, x_2; and at least one position gyro must be used in order to determine pitch angle, x_1. It was found that using two of each type of sensors resulted in approximate feedbacks quite close to the one desired. Three sensor stations were chosen at 25, 50, and 75 meters and various combinations of sensor locations were selected. That is, the six sensors were permitted between these three stations in a variety of ways. Not all least squares solutions turned out to be good, or in fact stable, although with six sensors good solutions resulted in two thirds of the cases. Three typical cases are presented in Table 7. Here y_1 and y_2 are accelerometer locations, y_3 and y_4 rate gyro locations, and y_5 and y_6 position gyro locations. The resulting gains $k_1,k_2,\ldots,k_9$ are of course components of the vector $\gamma_1 k_1 + \gamma_2 k_2,+\ldots+ \gamma_9 k_9$. The desired gain vector $k_1,k_2,\ldots,k_9$ as listed in Table 5 case 4 is presented in Table 7 for comparison. The resulting closed loop characteristic roots are presented in Table 8. It is noted that in two cases the closed loop is somewhat tighter than desired, while in the other case the closed loop is not stable. The calculations performed tacitly assume that the controller is capable of responding to the signal $k_1 x_1 +\ldots+ k_9 x_9$. Thus, it may be that too tight a controller results in gimbal angle saturation. This question was not investigated. Neither was the question of gust response. Finally it is stated that the sensor stations were selected completely arbitrarily and several other selections of stations worked equally as well.

CONCLUSIONS

A linear controller for a typical rocket booster was designed by use of the quadratic penalty function criterion and its approximate implementation was accomplished using less sensors than the plant order. It is believed the procedures involved constitute a design method which is capable of better realization of optimal linear controllers for flexible vehicles. Such techniques together with the usual techniques of filtering very high frequency dynamics out of the sensor signals yield an attack to the synthesis problem for control of a very flexible vehicle.

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F	Thrust Force
X	Axial Air Force
N'	Air Force Perpendicular to Long Axis of Vehicle
R'	Control Force Perpendicular to Long Axis of Vehicle
m	Mass of Vehicle
z	Displacement of Mass Center of Vehicle Perpendicular to Standard Path
α	Angle of Attack
β	Swivel Motor Deflection or Vane Deflection
φ	Attitude Angle
V_o	Velocity of Vehicle Along its Path
η_i	Normalized Flexure Modes
d_i	Constants
c_1	A constant
c_2	A Constant
e_i	Constants
f_i	Constants
ζ_i	Flexure Mode Damping Ratio
ω_i	Flexure Mode Frequency
$Y_i(y)$	Normalized Flexure Mode Shape As A Function of y the Distance From The Tail
y_β	Engine Pivot Point
y_{cg}	Position of Mass Center With Respect to Vehicle Tail
y	A Parameter Measuring Distance From the Tail of the Vehicle

TABLE 1

LIST OF DEFINITIONS

$$A = \begin{bmatrix} 0 & 1.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .2165 & -.0356 & 0 & -.0299 & 0 & -.0270 & 0 \\ -.0458 & 1.000 & -.0133 & .0004 & 0 & .0006 & 0 & .0007 & 0 \\ 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -29.81 & -.0546 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -169.0 & -.1300 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -334.3 & -.1828 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 0 & -1.138 & -.0348 & 0 & 29.56 & 0 & 47.25 & 0 & 16.40 \end{bmatrix}$$

TABLE 2

DYNAMICAL CONSTANTS FOR A TYPICAL FLEXIBLE VEHICLE

λ_1 = .046	λ_2 = .4333	λ_3 = -.493
λ_4 = -.270 + 5.460i	λ_5 =-.0270-5.460i	λ_6 =-.0650+12991
λ_7 = -.0650-12.99i	λ_3 =-.0910+18.28i	λ_9 =-.091-18.28i

TABLE 3

CHARACTERISTIC ROOTS OF THE OPEN LOOP SYSTEM

	$q_1^{(1)}$	$q_2^{(1)}$	$q_3^{(1)}$	$\lambda_1^{(1)}$	$\lambda_2^{(1)}$	$\lambda_3^{(1)}$	$k_1^{(1)}$	$k_2^{(1)}$	$k_3^{(1)}$
Case 1	2	2	2	-.035	$-1.378 + .62301i$	$-1.378 - .62301i$	2.508	2.450	-.2720
Case 2	.5	.5	.5	-.036	$-.922 + .5561i$	$-.922 - .5561i$	1.280	1.641	-.0470
Case 3	.1	1.0	.1	-.004	-.500	-1.216	.6290	1.497	.0710
Case 4	.1	.05	.5	-.043	$-.761 + .5741i$	$-.761 - .5741i$	1.096	1.365	-.078

TABLE 4

RIGID BODY CONTROLLERS FOR SEVERAL CHOICES OF THE WEIGHTING MATRIX $Q^{(1)}$.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
Case 1	-.0427	-.7700 +.588i	-.770 -.588i	-1.486 +5.28i	-1.486 -5.28i	-2.283 +12.77i	-2.283 -12.77i	-.726 +18.25i	-.726 -18.25i
Case 2	-.0427	-.772 +.588i	-.772 -.588i	-.474 +5.44i	-.474 -5.44i	-.749 +12.98i	-.749 -12.89i	-.272 +18.28i	-.272 -18.28i
Case 3	-.0427	-.773 +.587i	-.773 -.587i	-.157 +5.46i	-.157 -5.46i	-.246 +13.00	-.246 -13.00i	-.121 +18.28i	-.121 -18.28i

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
Case 1	1.099	1.464	-.0515	-.0625	-.1007	-.3450	-.0938	-1.0134	-.0682
Case 2	1.099	1.374	-.0785	-.0375	-.0311	-.0662	-.0288	-.1162	-.0216
Case 3	1.099	1.345	-.0865	-.0145	-.0100	-.0143	-.0078	-.0094	-.0037

$q_1 = .1 \quad q_2 = .05 \quad q_3 = .5$

$q_j = W, \; j = 4,5,\ldots,9$

Case 1: W = .01   Case 2: W = .001   Case 3: W = .0001

TABLE 5

CLOSED LOOP CHARACTERISTIC ROOTS AND FEEDBACK GAINS FOR SEVERAL WEIGHTING MATRICES APPLIED TO A NINTH ORDER PLANT

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9
Case 1	-.0427	-.760 +.574i	-.760 -.574i						
Case 2	-.0427	-.770 +.585i	-.770 +5.46i	-.152 -5.46i					
Case 3	-.0427	-.772 +.587i	-.772 -.587i	-.153 +5.46i	-.153 -5.46i	-.246 +13.0i	-.246 -13.0i		
Case 4	-.0427	-.773 +.587i	-.773 -.587i	-.157 +5.46i	-.157 +5.46i	-.246 +13.0i	-.246 -13.0i	-.121 +18.28i	-.121 -18.28i

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
Case 1	1.096	1.365	-.0778						
Case 2	1.096	1.346	-.0842	-.0145	-.0097				
Case 3	1.099	1.345	-.0860	-.0140	-.0097	-.0145	-.0078		
Case 4	1.099	1.345	-.0865	-.0145	-.0100	-.0143	-.0078	-.0094	-.0037

Case 1: Rigid Body - Third Order Plant
Case 2: Rigid Body +1 Flexure Mode - Fifth Order Plant
Case 3: Rigid Body +2 Flexure Modes - Seventh Order Plant
Case 4: Rigid Body +3 Flexure Modes - Ninth Order Plant

$q_1 = .1 \quad q_2 = .05 \quad q_3 = .5$

$q_j = .0001 \quad j > 3$

TABLE 6

CHARACTERISTIC ROOTS AND GAINS FOR A SERIES OF PROBLEMS OF CONTROLLER DESIGN WITH INCREASING PLANT ORDER

CASE 1	13.492	1.3302 1.2120	-12.016 -.4243 11.641						
CASE 2	13.524	1.3020 1.1703	-12.063 -.41726 11.691	-.01074 -.0125 .00111 .00869	-.00119 -.00106 .00054 .00029				
CASE 3	13.541	1.2956 1.1610	-12.082 -.41553 11.710	-.01075 -.012507 .001108 .00872	-.00120 -.001083 .000535 .000013 .000289	-.008597 -.010426 .000725 .000100 .000088 .027456	-.000178 -.000138 .000096 -.000012 -.000000 .000001 .000162	(All Matrices are Symmetric)	
CASE 4	13.548	1.2951 1.1598	-12.089 -.41554 11.716	-.011071 -.012897 .001135 .009015	-.001196 -.001078 .000531 .000017 .000299	-.008633 -.010476 .000722 .000105 .000092 .027603	-.000180 -.000180 .000095 -.000013 .000000 .000002 .000162	-.004266 -.004266 .000542 .000049 .000039 .010012 .000056 .073434	-.000098 -.000068 .000052 -.000002 -.000000 -.000028 -.000000 -.000000 .000220

TABLE 7

P MATRICES FOR A SERIES OF PROBLEMS OF CONTROLLER DESIGN WITH INCREASING PLANT ORDER

Desired Feedback

k_1=1.099	k_2=1.345	k_3=-.0865	k_4=-.0145	k_5=-.0100	k_6=-.0143	k_7=-.0078	k_8=-.0094	k_9=-.0037

CASE 1

y_1 = 50	y_2 = 25	y_3 = 25	y_4 = 50	y_5 = 50	y_6 = 25			
γ_1=-.0028	γ_2=.0028	γ_3=1.119	γ_4=.2248	γ_5=.2570	γ_6=.8412			
k_1=1.098	k_2=1.344	k_3=-.0860	k_4=-.0348	k_5=-.0386	k_6=-.0049	k_7=.0079	k_8=-.0123	k_9=.0172

CASE 2

y_1 = 50	y_2 = 25	y_3 = 25	y_4 = 75	y_5 = 50	y_6 = 25			
γ_1=-.0028	γ_2=-.0028	γ_3=1.176	γ_4=.1685	γ_5=.2570	γ_6=.8413			
k_1=1.098	k_2=1.345	k_3=-.086	k_4=-.040	k_5=-.0256	k_6=-.0025	k_7=-.0125	k_8=-.0131	-.0135

CASE 3

y_1 = 50	y_2 = 25	y_3 = 25	y_4 = 75	y_5 = 75	y_6 = 25			
γ_1=-.0030	γ_2=+.0030	γ_3=1.760	γ_4=.1685	γ_5=.2157	γ_6=.8832			
k_1=1.099	k_2=1.345	k_3=-.086	k_4=-.0190	k_5=-.0256	k_6=-.006	k_7=-.0256	k_8=-.0131	k_9=-.0135

TABLE 8

SENSOR GAINS AND RESULTING FEEDBACK USING SIX SENSORS

CASE 1

$\gamma_1 = -.043$	$\gamma_2 = -.769+.601i$	$\gamma_3 = -.769-.601i$
$\gamma_4 = -.572-536i$	$\gamma_5 = -.572+536i$	$\gamma_6 = .119+13.0i$
$\gamma_7 = .119-13.0i$	$\gamma_8 = .049-183.i$	$\gamma_9 = .049+18.3i$

CASE 2

$\gamma_1 = -.043$	$\gamma_2 = -.765+.598i$	$\gamma_3 = -.765-.598i$
$\gamma_4 = -.401-545i$	$\gamma_5 = -.401+5.45i$	$\gamma_6 = -.352+12.9i$
$\gamma_7 = -.352-12.9i$	$\gamma_8 = -.199-18.3i$	$\gamma_9 = -.199+18.3i$

CASE 3

$\gamma_1 = -.043$	$\gamma_2 = -.782+.596i$	$\gamma_3 = -.782-.596i$
$\gamma_4 = -.386-5.39i$	$\gamma_5 = -.387+5.39i$	$\gamma_6 = -.352+12.9i$
$\gamma_7 = -.352-12.9i$	$\gamma_8 = -.199+18.3i$	$\gamma_9 = -.199-18.3i$

TABLE 9

CLOSED LOOP CHARACTERISTIC ROOTS FOR THE

APPROXIMATE FEEDBACKS LISTED IN TABLE 7